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Rupture of Heterogeneous Media in the Limit of Infinite Disorder

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We consider a random fuse or random fragile element model. We show that, in the limit of infinite disorder in the bond-breaking thresholds, the rupture of a lattice is a "disguised" percolation process. Therefore, just before the final overall rupture of the lattice, we obtain scaling relations of various physical properties as a function of the number of bonds broken.

KEY WORDS: Fracture; percolation; heterogeneous media.

The rupture properties of disordered systems are of considerable interest both from the technological and from the scientific point of view. In parallel to the mechanical problem of rupture (see, e.g., ref. 1), recent advances have been made in the scalar case of electrical conduction, i.e., the breakdown of fuse networks.[2–5]

Three main approaches can be considered (see, e.g., ref. 6):

1. The *random threshold* problem,[2] where a lattice consists of fuses having the same conductance, but randomly distributed breaking threshold voltages v_b.
2. The *random conductance* case, where the conductances have random values, but the breaking thresholds are identical.[5]
3. The *random fuse* problem, where the disorder is induced only by a random initial depletion of the lattice as in a percolation process.[7]

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In this paper, we focus exclusively on the first case, which is still also under numerical study both in the original electrical case[2] and in a mechanical version.[8] In the following, we refer to current and voltage, although one can transpose the results directly into a mechanical framework by using force and displacement instead.

Let us first recall the procedure used in the random threshold model: One starts with a full lattice and applies a voltage drop V across it. As the voltage is increased, one bond will reach its limit voltage and then burn out, thus changing its conductance to zero. The same procedure is repeated again with the resulting lattice, until finally the lattice is broken apart and thus becomes insulating. Of course, in practice one solves the distribution of potentials in the lattice for a unit voltage drop $V=1$ and looks for the maximum value of the ratio

$$\Lambda_i = |v_i|/v_{bi} \tag{1}$$

over all bonds; v_i is the voltage drop in the ith bond for $V=1$, and v_{bi} is the ith breaking voltage. If the maximum Λ is reached on bond j, then this bond is broken when the external voltage is $V=1/\Lambda_j$.

Two elements are important in determining the rupture sequence and can be used to define two limiting cases:

(a) The ratio Λ is controlled by the $|v_i|$; this corresponds to the *small-disorder* limit in the distribution of thresholds, or to a large distribution of intensities in the lattice.

(b) The ratio Λ is controlled by the v_{bi}; this corresponds to the *large-disorder* limit in the breaking voltage compared with the distribution of currents in the lattice. We will concentrate upon this latter case in the rest of this paper. It is typical for the beginning of a rupture process (local microcracks), whereas the small-disorder limit is rather reached at the end of the rupture sequence (development of a single macrocrack). In most situations, one crosses over from the large-disorder regime to the small-disorder limit abruptly as rupture develops.[6]

In case (b), we can neglect the width of the voltage distribution. However, we should distinguish clearly between two classes of bonds: those that carry a current (they constitute the "backbone," by definition), and those that do not belong to the backbone (they are part of either dangling ends or isolated clusters). Our hypothesis of infinite disorder results in deciding at random which bond will break, and this with equal probability for each bond in the backbone. Obviously, all bonds that do not belong to the backbone carry no current and thus cannot be burned out. It thus turns out that the rupture is determined in this case by the geometry alone

(the backbone can also be defined in a purely geometrical way, as the intersection of all self-avoiding walks from one electrode to the other) rather than by transport properties (i.e., solving the complete current distribution).

This problem is now very close to a percolation problem. In the latter case, one would cut bonds at random in the lattice. Let us call p the proportion of bonds present at a given stage of the depletion. With our rule of breaking, contrary to the percolation process, we have to check at each step whether the bond we have picked at random does or does not belong to the backbone. If it does not belong to the backbone, then we select another one, until finally we pick one bond on the backbone and thus cut it. Let us assume that we use the same random choice of bonds to be removed in parallel in both problems. Then, for a given p in the percolation problem, we can associate the proportion of bonds $r(p)$ that have not been burnt out in the fuse problem. Some properties of the lattice will not be affected by the difference in the two processes; for instance, their backbones will be identical. Asymptotically, when the lattice size tends to infinity, there is a unique and well-defined correspondence between r and p (see Fig. 1). This correspondence yields information (and scaling relations) about the rupture problem from percolation.

The reason for introducing both parameters r and p is that p is the natural percolation parameter, but unfortunately, it is not defined experimentally. On the contrary, r is observable: it is the proportion of bonds effectively broken, and therefore is related to the "damage parameter" often used in the literature. The function $r(p)$ allows a direct translation of a physical feature into a practical parameter used in the percolation model.

The relation between r and p is a purely geometrical problem, which can be solved numerically by using an algorithm to identify the backbone. This problem is known to be difficult.[9] However, we can simplify it considerably in the two-dimensional case by considering the dual lattice: To each cell is associated a site in the dual lattice whose bonds are such that they cross one and only one bond in the original lattice (see Fig. 2). The bonds that were conducting are associated with resistors, and the insulating bonds (the fuses that have burned out) become superconductors in the dual lattice. Therefore, the "dual-fuse" is a resistive element that becomes short-circuited whenever the current flowing through it is larger than the threshold v_b (equal to the voltage threshold for the original fuse). The boundary conditions also have to be changed accordingly: An insulating side becomes an equipotential and an equipotential is turned into an insulating border. Both problems are equivalent. However, numerically, the second one is easier to handle. The reason for this is that the dual of the

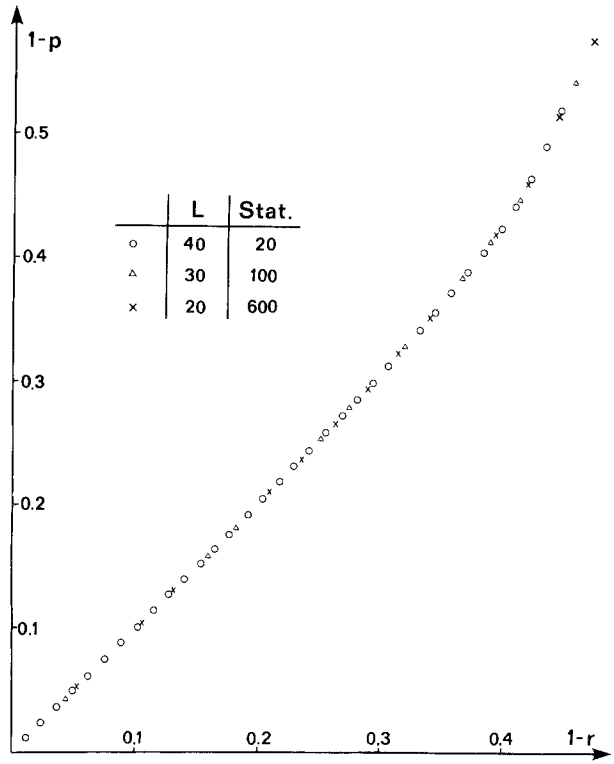

Fig. 1. Relation between the number of bonds burned out in our model $1-r$ versus the proportion of bonds p present in the equivalent percolation problem. The difference between r and p is to be attributed to the bonds that were not burned out in the breakdown model because they were not carrying any current (they were not part of the backbone).

backbone is now the set of bonds that are not short-circuited by a superconducting cluster. We note that this is the basis of a very efficient method to generate percolation backbones.[10]

Figure 1 represents the relation between r and p obtained through the numerical technique mentioned above. Two domains of this curve are of particular interest: p close to 1, and p close to the percolation threshold.

We generated respectively 600, 100, and 20 lattices of size 20, 30, and 40 and recorded the mean value of $r(p)$ averaged over all samples that were still not broken. (This is why the curve goes down to values of p below the percolation threshold $p_c = 0.5$ in our case.) The intersection of the curve $r(p)$ with the value $p = p_c$ gives a first estimate of the proportion of intact bonds at rupture: $r_b = 0.56 \pm 0.01$. We also obtained a direct estimate of this value by averaging the value of r_b for each lattice. This gave $r_b =$

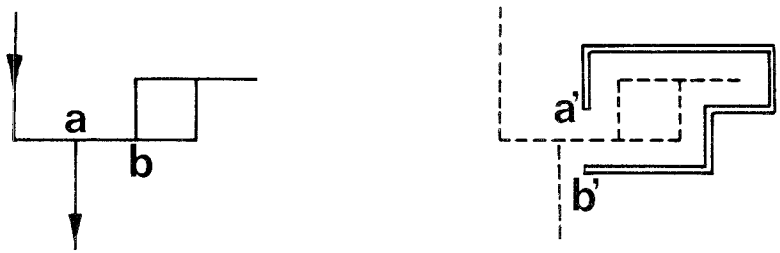

Fig. 2. The original lattice is made up of fuses with randomly distributed thresholds and uniform conductance. The dual lattice consists of bonds that become superconductors when the voltage to which they have been submitted is larger than the threshold. We begin with a full lattice and progressively increase the voltage until one bond burns out and we repeat the process until the whole lattice breaks apart. This process, in the limit of infinite disorder of bond strengths, turns into a disguised percolation process where bonds are removed at random whenever they belong to the current-carrying part of the lattice (the backbone). In the original lattice shown on the left, the bond A–B cannot be removed because no current flows through it; in the dual lattice, represented on the right, the conjugated bond is A′–B′, which is short-circuited by a cluster of superconducting bonds.

0.564 ± 0.001. We obtained the same estimate for all three sizes from 20 to 40. It is remarkable how close r is to p for a very wide range of values; even close to the breaking point, r_b is rather close to p_c (only 10% difference), indicating that the influence of dead ends or finite clusters should be taken into account only in the immediate neighborhood of the threshold.

At the beginning of the rupture process—p close to 1—almost no bond is isolated and therefore r is equal to p. Moreover, we know that, in the domain of a large defect dilution (small damage), self-consistent techniques give very accurate results on the transport properties of these systems as a function of p. For instance, on a square lattice, we can write the conductivity $G(p)$ as

$$G(p) = G_0(2p - 1) \tag{2}$$

where G_0 is the conductivity of the undamaged lattice.[11] This expression is valid up to the fourth order in $(1 - p)$.

If we call $f(x)$ the distribution of probability of the breaking voltages v_b, defined in the interval $[0, 1]$, then, for a given value of p, the mean value of v_b for the bond that breaks is $v_b = y(p)$ such that

$$\int_0^{y(p)} f(x)\, dx = 1 - p \tag{3}$$

and thus the voltage threshold for the entire lattice $V_b(p)$, as can be seen from Eq. (1), varies as

$$V_b(p) = y(p)/\langle v(p) \rangle \tag{4}$$

where $\langle v \rangle$ is the *mean* voltage drop in a bond for a unit external voltage drop. Our hypothesis of infinite disorder will be valid until the end of the breaking process only in the case where the distribution of $1/v_b$ is much wider than the distribution of currents in the lattice. We know, however, that the distribution of current will become infinitely large at the percolation threshold. Thus we will observe a crossover to another regime, controlled rather by the geometry than by the distribution of threshold voltages, whenever the width of the two distributions match.

However, if the function f has a singularity at 0, $f(x) \propto x^\alpha$ and if α tends to -1, the rupture will always be controlled by the distribution of threshold voltages (a similar result has been suggested through numerical simulation in "model III" (our model 3) of ref. 6). It is the situation of infinite disorder. For narrower distributions, the hypothesis of infinite disorder will be valid at the beginning of the process and one will then cross over to another regime during the breaking process as the distribution of voltage drops broadens.

As we consider the case of "infinite disorder," we can neglect in Eq. (4) the variation of $\langle v \rangle$ with p and thus identify the variation of the external breaking voltage with the one of y given in Eq. (3). Since we also know the conductivity, Eq. (1), we can write down the current flowing through the lattice at rupture:

$$I_b(p) = G(p)\, V_b(p) = (G_0/\langle v \rangle)(2p-1)\, y(p) \tag{5}$$

At the beginning of the rupture process, where the relation (2) holds, we expect r to be close to p. Figure 1 shows indeed that r and p are indistinguishable for p greater than 70%. So, we can change p into r in the previous equations without losing much precision.

We also get some other information about the relation between r and p close to the percolation point $p = p_c$ if we note that the derivative dr/dp is nothing but the probability that a bond belongs to the backbone $P_{bb}(p)$ for a given value of the control parameter p. From percolation theory, we know that the behavior of $P_{bb}(p)$ is singular at the threshold:

$$P_{bb}(p) \propto (p - p_c)^{\beta_{bb}} \tag{6}$$

where β_{bb} is simply related to the fractal dimension of the backbone d_{bb} by[12]

$$\beta_{bb} = \nu(d - d_{bb}) \tag{7}$$

where ν is the correlation length critical exponent and d the space dimension. For instance, in two dimensions, $\nu = 4/3$, $d_{bb} \approx 1.62$, $\beta_{bb} \approx 0.51$; in three dimensions, $\nu \approx 0.89$, $d_{bb} \approx 1.77$, $\beta_{bb} \approx 1.1$.[9] Therefore,

$$dr/dp \propto (p - p_c)^{\beta_{bb}} \tag{8}$$

Hence

$$(p - p_c) \propto (r - r_b)^{1/(1+\beta_{bb})} \tag{9}$$

where r_b is the proportion of present bonds at the final rupture of the lattice. This equation stresses the connection between the rupture problem and the results of the theory of percolation. However, it had not been possible to verify numerically this relation in Fig. 1. It should hold only in the close vicinity of the threshold and apparently, for the lattice sizes considered here, this behavior is masked by finite-size corrections.

As an example, we know that the conductance G of the percolation lattice at the percolation threshold varies critically with an index t:

$$G \propto (p - p_c)^t \tag{10}$$

where $t \approx 1.29$ in two dimensions and $t \approx 2.0$ in three dimensions. Thus,

$$G \propto (r - r_b)^{t/(1+\beta_{bb})} \tag{11}$$

Some other properties of percolation can also be used directly: The distribution of cluster sizes in the dual lattice in two dimensions, i.e., the number of sites that are connected in the dual lattice by superconducting bonds, at the rupture point is identical to the distribution observed in percolation at threshold, since, in our model of rupture, one does not remove bonds that are already short-circuited by a superconducting cluster; therefore the connectivity of the sites in these clusters is not affected. If one defines the clusters as the number of *bonds* (and not sites) that are connected by a superconducting path, then there is apparently a large difference between percolation and the rupture case. However, if we note that the clusters in the rupture problem are loop-free (in the dual lattice) (see Fig. 2), then we can relate the number of bonds b in the clusters to the number of sites s by $b = s - 1$. Therefore, both the site and bond numbering will produce a distribution of cluster sizes identical to the one encountered in percolation. For similar reasons, the fractal dimension of the main crack will be that of the infinite cluster in percolation.

In conclusion, we have shown that the final rupture of infinitely disordered media is related to a percolation phenomenon, and thus constitutes a critical point. In a real material, disorder will be "finite" and therefore there will be a crossover between the *disorder-controlled* regime and the *brittle* regime (development of a macrocrack). This crossover will happen at a finite correlation length, given by the corresponding percolation problem. In this sense, the rupture process can be seen as a critical phenomenon where the width of the distribution of breaking thresholds acts as a control parameter.

Our results are certainly easy to check experimentally and it would be interesting to do so to know if the concept of infinite disorder is of relevance to any experimental situations and up to what point one can neglect of the large distribution of currents that will appear just before rupture.

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REFERENCES

1. *Fragmentation, Form and Flow*, Annals of the Israeli Physical Society, Vol. 8 (1986), and references cited therein.
2. G. G. Batrouni, K. Khang, S. Redner, L. de Arcangelis, and H. J. Herrmann, *Phys. Rev. B*, to appear.
3. P. M. Duxbury, P. D. Beale, and P. L. Leath, *Phys. Rev. Lett.* **57**:1052 (1986).
4. A. Gilabert, C. Vanneste, D. Sornette, and E. Guyon, *J. Phys. (Paris)* **48**:763 (1987).
5. H. Takayasu, *Phys. Rev. Lett.* **54**:1099 (1985).
6. M. Sahimi and J. D. Goddard, *Phys. Rev. B* **33**:7848 (1986).
7. L. de Arcangelis, S. Redner, and H. J. Herrmann, *J. Phys. Lett. (Paris)* **46**:585 (1985).
8. A. Hansen, H. J. Herrmann, and S. Roux, preprint.
9. H. J. Herrmann, D. C. Hong, and H. E. Stanley, *J. Phys. A* **17**:L261 (1984).
10. S. Roux and A. Hansen, *J. Phys. A* **20**:L1281 (1987).
11. S. Kirkpatrick, *Rev. Mod. Phys.* **45**:574 (1973).
12. S. Kirkpatrick, in *Electrical Transport and Optical Properties of Inhomogeneous Media*, J. C. Garland and D. B. Tanner, eds. (AIP, New York, 1978), p. 99.